Introduction: Basic concepts. Passive elements.

Contents

1 Basic concepts

Electronics deals with voltage and current interaction in a network of resistances R, capacitances C, inductances L and active elements such as transistors. The main purpose of electronics circuits is to amplify signals or to produce signals of a desired waveform.

We will start off with a very theoretical introduction of the laws of electrodynamics to provide a sound theoretical background. Nevertheless in most situations where physicists have to deal with electronics, the knowledge of Ohm's law and circuit theory is sufficient.

1.1 Maxwell Equations and Lorentz Force

Maxwell's equations are a set of four equations that describe the behaviour of electric and magnetic fields and their interactions with matter. The four equations are given below in differential form (no movement of charges):

1st law (*Gauss' law for electricity*): The electric flux out of any closed surface is proportional to the total charge enclosed within the surface.

$$\nabla \cdot \vec{D} = \rho$$

with

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

where $\vec{D}$ is the electric displacement field [C/m^2], ρ the free electric charge density [C/m^3], $\vec{E}$ the electric field strength [V/m], ϵ_0 the permittivity of free space [F/m] and $\vec{P}$ the electric polarisation [C/m^2].

2nd law (*Gauss' law for magnetism*): The net magnetic flux out of any closed surface is zero.

$$\nabla \cdot \vec{B} = 0$$

where $\vec{B}$ is the magnetic flux density [Vs/m^2].

3rd law (*Faraday's law of induction*): The line integral of the electric field around a closed loop is proportional to the rate of change of the magnetic flux though the area enclosed by the loop.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

4th law (*Ampere's law*): The line integral of the magnetic field around a closed loop is proportional to the electric current flowing through the loop.

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

with

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

where $\vec{H}$ is the magnetic field strength [A/m], $\vec{J}$ the current density [A/m^2] ($\vec{J} = \vec{I}/A$, the current $\vec{I}$ on a surface A), μ_0 the magnetic permeability of free space [Vs/(Am)] and $\vec{M}$ the magnetisation of the material [A/m].

The force F acting on a charged particle (charge q, velocity $\vec{v}$) moving in the presence of an electric and a magnetic field is called **Lorentz force**:

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

1.1.1 Electrostatics

Electric fields are produced by stationary charges. The force in vacuum between two charges between q_1 and q_2 is given by Coulomb's Law:

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}\ [N]$$

where $\epsilon_0 = 8.854 \times 10^{-12}$ F/m and r corresponds to the distance between the 2 charges. This expression was determined experimentally by Coulomb in 1785.
The electric field generated by a single charge is:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}\ [V/m]$$

In this definition the electric field points from the positive towards the negative charge.

1.1.2 Magnetostatics

Magnetic fields are produced by moving charges. The strength of the magnetic field from an infinitesimal section of a wire is

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{l} \times \hat{r}}{r^2}$$

with μ_0 = $4\pi \times 10^{-7}$ [N/A]. This law was found experimentally in 1820 by Biot-Savart.

1.2 Voltage

In case there is no varying magnetic field, the curl of the electric field vanishes so that we can define the electric field as the gradient of a potential ϕ

$$\vec{E} = -\nabla\phi$$

Given this definition, the potential from a point charge is

$$\phi = \frac{q}{4\pi\epsilon_0 r}\ [V]$$

Since the electric field is the gradient of the potential, the potential difference between two points can be found by integrating the electric field along an arbitrary path between the points.

$$V(\vec{x},\vec{y}) = \phi(\vec{y}) - \phi(\vec{x}) = -\int_{\vec{x}}^{\vec{y}} \vec{E}\cdot d\vec{l}\,[V]$$

We can also note the relation between voltage and work W: $dW = Fdl = QEdl$. Therefore $dV = dW/Q$.

1.3 Current

Current I , measured in Amperes [A], is the time rate at which charges Q pass a given reference point. Thus

$$I = \frac{dQ}{dt}$$

1.4 Power

If we take the expression for power (rate of doing work) and multiply it by dQ/dQ we obtain:

$$P = \frac{dW}{dt} = \frac{dW}{dt}\frac{dQ}{dQ} = \frac{dW}{dQ}\frac{dQ}{dt} = VI\ [W]$$

Hence power is voltage multiplied by current.

1.5 Ohm’s Law

To make a current flow it is needed to push on the charges. How fast they move depends on the nature of the material. For most substances, the current density $\vec{J}$ is proportional to the force per unit charge:

$$\vec{J} = \sigma\vec{F}$$

The proportionality constant (σ) is called conductivity, and its reciprocal is the **resistivity** $\rho = 1/\sigma$. Microscopically we can understand the resistivity. In some media, in the presence of an electrical field, an electron is accelerated. It then moves freely until it collides with one of the atoms of the material that slows it down, after which the electron gets accelerated again by the electric field. Resistivity (ρ) is a material property that relates to the average interval between collisions. Developing the previous expression for the electromagnetic force (see Lorentz force), we obtain

$$\vec{J} = \sigma(\vec{E} + \vec{v}\times\vec{B})$$

Normally, the velocity of the charges is sufficiently small that the second term can be ignored, yielding

$$\vec{J} = \sigma\vec{E}$$

This last expression is known as **Ohm's Law**. Beware, the physics is in the relationship between current density and the force. Ohm's Law is just a special case.

Let's imagine a wire with cross-sectional area A and length l made of a material of conductivity σ. What current flows? The current can be written as $I = JA$ and taking into account Ohm's Law and the relation $E = V/l$:

$$I = JA = \sigma EA = \frac{\sigma A}{l} V$$

This example shows us that the current that flows is proportional to the voltage difference. The proportionality constant depends on the conductivity of the material and also its geometry. We will call this constant **resistance** (R), allowing us to express Ohm's law in the more common way:

$$V = RI$$

Resistance is measured in Ohm (Ω).

1.6 Joule's Heating Law

When current flows in a material, the repeated collisions of the electrons with the atoms transfer energy to the atoms with the result that the temperature of the material increases. A resistor can therefore be considered as an energy-transforming device: it converts electrical energy into heat. The amount of energy conversion by a resistor can be expressed by:

$$P = VI = I^2 R = \frac{V^2}{I}$$

This expression is known as **Joule's Law** where the power is expressed in $Watt$. Integrating we obtain the thermal energy (W) dissipated in a time interval t:

$$W = I^2 Rt$$

This is known as **Joule's heating law**.

1.7 Circuit Theory

1.7.1 Kirchhoff's Laws

Many circuits are extremely complex and cannot be solved applying simply Ohm's Law. Kirchhoff's Laws are a powerful set of laws which enable one to analyse arbitrary circuits e.g. with many power sources and branches.

1. <u>1st Kirchhoff's law</u>: At a node (point at which two or more circuit elements have a connection) the algebraic sum of the currents entering is zero (charge conservation).

   $$\sum I_n = 0$$

   Choosing e.g. node C in fig. 1, one can therefore note that the incoming currents equal the outgoing current: $I1 + I2 = I3$.

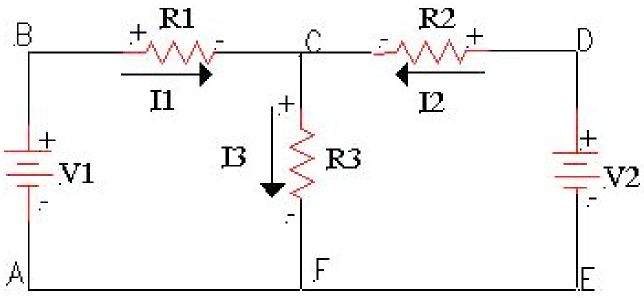


Figure 1: Circuit with 2 branches.

2. 2nd Kirchhoff's law: The algebraic sum of the voltages around any closed path in a circuit is zero (energy conservation).

$$\sum V_n = 0$$

Looking again at fig. 1, we can write down the 2 equations for the 2 branches:

$$V1 = R1I1 + R3I3$$

$$V2 = R2I2 + R3I3$$

From these laws we can state that circuit theory is a linear theory.

1.7.2 Equivalence

Two circuits are equivalent if they have the same $I - V$ characteristics at their terminal.

1.7.3 Superposition

This powerful theorem comes directly from linearity. In a network with multiple voltage sources, the current in any branch is the sum of the currents which would flow in that branch due to each voltage source acting alone with all other voltage sources replaced by their internal impedances (see later for the definition of the impedance).

1.7.4 Thevenin's Theorem

Any circuit can be replaced by a series combination of an ideal source V_{th} and a resistance R_{th}, where V_{th} is the open-circuit voltage and R_{th} is the ratio of the open-circuit voltage with the short-circuit current.

Fig. 2 shows an example for a circuit with 2 voltage sources, 4 resistances, the output terminal AB and a load resistance $R2$. The part to the left of A we can replace with a Thevenin equivalent of the series combination of a voltage source $V_{th1} = V_{S1}\frac{R1}{R1+R4}$ and a series resistance $R_{th1} = \frac{R1R4}{R1+R4}$.
The part to the right of B can be replaced by a Thevenin equivalent of the series combination

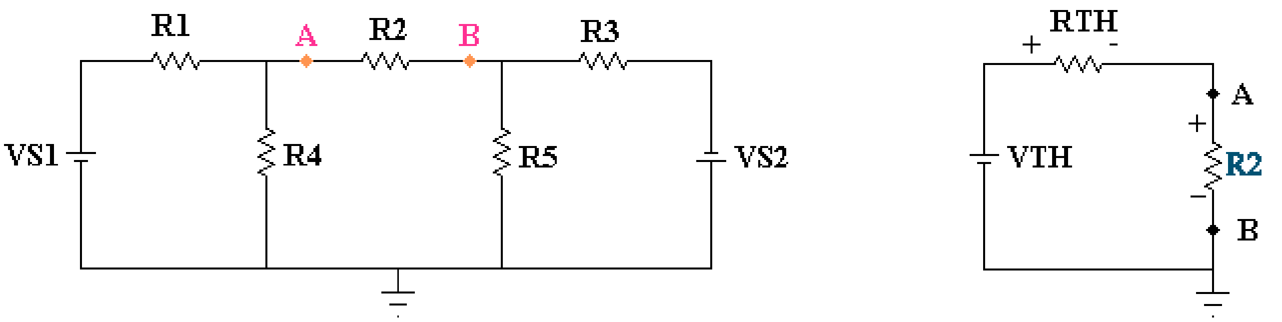


Figure 2: Circuit with 2 voltage sources and a load resistor $R2$ on the left and its Thevenin equivalent on the right.

of a voltage source $V_{th2} = V_{S2}\frac{R5}{R3+R5}$ and a series resistance $R_{th2} = \frac{R3R5}{R3+R5}$. Finally, these two parts can be combined into a series combination of a voltage source $V_{th} = V_{th1} + V_{th2}$ with the series resistance $R_{th} = R_{th1} + R_{th2}$. Now the output voltage can easily be written down as $V_{out} = \frac{R2}{R_{th}+R2}$.

1.7.5 Norton's Theorem

It's the dual of Thevenin's theorem. Any circuit can be substituted by a parallel combination of a current source I_n and a resistance R_n. The resistance is the same as R_{th} and I_n is obtained short-circuiting.

1.7.6 Maximum Power Transfer and Matching

How much power can a source deliver to a load (R_L) that is connected to the source? Let's reduce the source by it's Thevenin's equivalent:

$$P = I^2 R_L = \left(\frac{V}{R+R_L}\right)^2 R_L$$

To get the maximum we need to differentiate:

$$\frac{dP}{dR_L} = V^2\frac{(R+R_L)^2 - 2R_L(R+R_L)}{(R+R_L)^4} \to R_L = R$$

This is an important result that has to be taken into account in the design of new circuits. Maximum power transfer is given when the load resistance matches the circuit resistance.

2 Decibels

The difference in power (e.g. input power to output power of a circuit) can be expressed in decibels (dB).

$$dB \equiv 10\log\frac{P_2}{P_1} = 10\log\frac{V_2^2}{V_1^2} = 20\log\left|\frac{V_2}{V_1}\right|$$

where we have used that $P \propto V^2$.

3 Passive components

3.1 Resistors

Resistors are the most simple electronic devices. As we have already seen, each time a current flows in a material, there is a resistance, depending both on the intrinsic characteristics of the material as well as its geometry, that opposes the current flow. Resistors are components built to have a determined value of electrical resistance. The graphic symbol of the resistors is in figure 3:

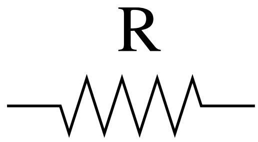


Figure 3: Schematic symbol of a resistor.

Current-Voltage relationship

The realtionship between current and voltage for this type of component is given by the Ohm's Law

$$V = IR$$

As R is a constant (it is assumed that R remains constant over a large range of voltages and temperatures), voltage and current in a resistor are in phase. This can be seen more easily if a sinusoidal voltage is applied to the resistor, as sketched in figure 4.

Instantaneous power

$$P = VI = I^2 R = \frac{V^2}{R} = \frac{V_p^2}{R} \sin^2 t$$

Average power

$$P_{ave} = \frac{1}{T} \int_0^T \frac{V^2}{R} dt = \frac{V_p^2}{2R}$$

Let us suppose now a resistor connected to a DC battery of voltage V. The power delivered to R would be constant with a value $P = VI = V^2/R$. Hence if we eqate DC power to the average AC power, we conclude that:

$$V = \frac{V_p}{\sqrt{2}} = 0.707 V_p$$

This is called the effective value of an AC voltage; a sinusoidal voltage of peak value V_p is as effective in delivering power to a resistor as a DC voltage of value $V_p/\sqrt{2}$.

Supplied energy

$$\omega_R = \int_0^t P\, dt' = \frac{V_p^2}{R} \int_0^t \sin^2 t\, dt' = \frac{V_p^2}{R} \left[t - \frac{\sin 2t}{2} \right]$$

This means that the energy supplied continuously increases, wiggling around the average value $V_p^2 t/2R$, that is equal to Joule's heating law.

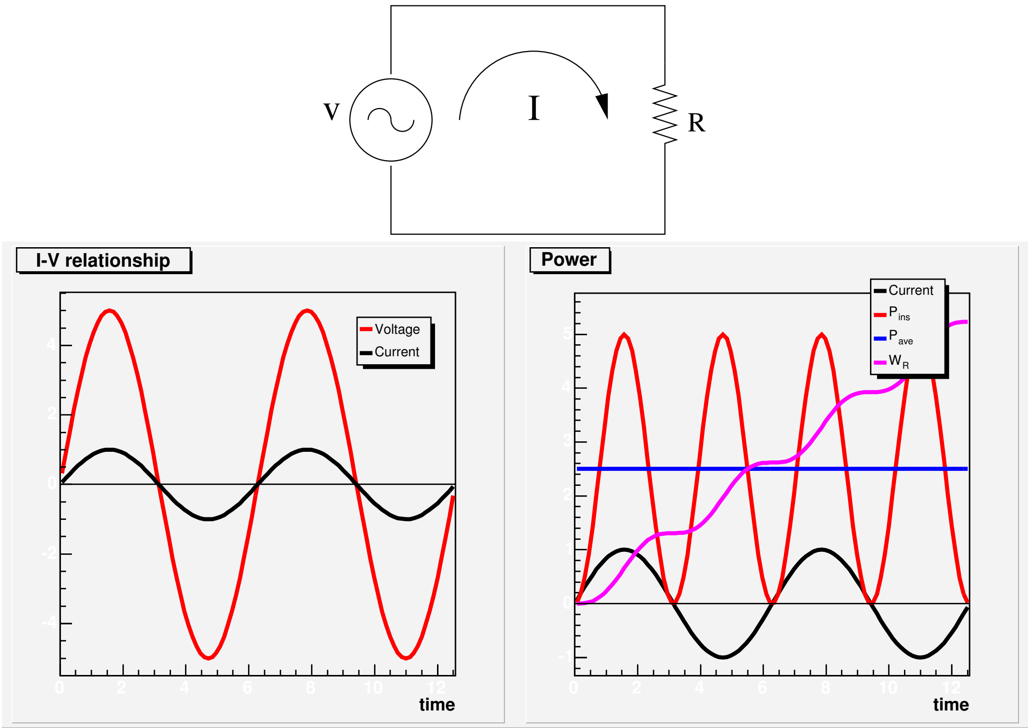


Figure 4: (a) Circuit diagram of a resistor with a voltage V applied. (b) A sinusoidal voltage results in an in-phase sinusoidal current in R. (c) Instantaneous power P in a resistor (always positive). Energy continues to increase with time.

3.1.1 Types of resistors

- Wire-Wound: Wire or ribbon wound as a single layer helix over a ceramic or fiberglass core. Residual inductance. Low noise and quite stable with temperature ($\pm$0.5-200 ppm/^oC) and R in [0.1,10^5] Ω.
- Metal Film: Printed circuits using a thin layer of resistance alloy on a flat or tubular insulating substrate. Characteristics similar to wire-wound resistors.
- Carbon Film: Similar in construction to axial lead metal film resistors. The carbon film is a granular material, therefore random noise may be developed because of variations in the voltage drop between granules. Can affect circuits with low signals.
- Carbon Composition: Cylinder of carbon-based resistive material molded into a cylinder. These resistors can have noise problems as the carbon films ones, but have been used for 50 years in electronics.
- Adjustable resistors: Cylindrical wire-wound resistors with a movable arm that is in contact with a resistive element. They are also called potentiometers. If a tool is needed they are called trimmers. They can vary linearly or logarithmically.
- Attenuators

3.2 Capacitors

A capacitor is a mechanical configuration that accumulates charge Q when a voltage V is applied and holds that charge when the voltage is removed. The proportionality constant between charge and voltage is the capacitance C, that is,

$$Q = CV$$

Capacitance is measured in $Farads$ (F). Most common values are in the range of pF and μF.
In case of two conducting plates $C = \epsilon A/l$. To obtain larger capacitances we can either increase the area (A), decrease the spacing (l) or increase the dielectric constant. The spacing imposes a limit on the voltage that can be applied, dictated by the dielectric breakdown strength of the insulating material used.

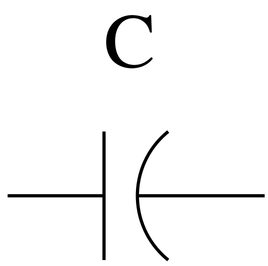


Figure 5: Schematic symbol of a capacitor.

Current-Voltage relationship

Derivating wrt time the last equation:

$$\frac{dQ}{dt} = C\frac{dV}{dt} \rightarrow I = C\frac{dV}{dt} = V_pC\cos t = V_pCsin(t + \pi/2)$$

This expression shows us that **a constant voltage across a capacitor produces no current** through the capacitor. Also we see that current and voltage are not in phase.

Instantaneous and average power

The instantaneous power is given by

$$P = VI = CV\frac{dV}{dt} = \frac{CV_p^2}{2}\sin 2t$$

The positive and negative values of P imply that power flows back and forth at twice the frequency of the applied voltage; first from source to capacitor ($P > 0$) and then from capacitor to source ($P < 0$) with an average power $P_{ave} = 0$. Therefore a capacitor does not consume energy. It simply stores it for a quarter period, and during the next quarter period it gives the energy back.

3.2.1 Energy stored

$$\omega_C = \int_0^t P\ dt' = \frac{1}{2}CV^2 = \frac{CV_p^2}{2}\sin^2 t = \frac{CV_p^2}{4}(1-\cos 2t)$$

This means that the average stored energy is a constant:

$$\omega_{ave} = \frac{1}{T}\int_0^T \omega_C dt = \frac{1}{T}\frac{CV_p^2}{4}\int_0^T (1-\cos 2t)dt = \frac{CV_p^2}{4}$$

Types of capacitors

Capacitors can be classified as below:

Mica capacitors Mica capacitors are built with a dielectric ($\epsilon_r \sim 6$) between two electrodes (usually silver). The metal and mica foils can be piled and each electrode connected to even and odd metal foils.

Mica capacitors have capacitances between pF and some tens of nF with tolerances ∼1%, and maximal voltage differences between 100 V and few thousand Volts. The dependence on temperature is quite high (∼100 ppm). Their response to high frequencies is quite good, therefore they are used in radiofrequency applications.

Film capacitors Capacitors where the dielectric is made of a plastic foil. Plastic is used because it can be manufactured with minimal imperfections in thin films. These capacitors are built by winding four layers of metal-plastic-metal-plastic into a cylinder and connecting both metal layers to the electrodes. Another technique is to use two layers of metallised foils (only on one side) that allows a higher capacitance in less volume. Usual materials are polyester (mylar), polystyrene, polycarbonate, polypropylene, polyfenylsulphite or teflon.

Film capacitors have capacitances between few hundreds pF and few tens of μF, with tolerances in the range of 1% and 20%, and maximal tensions between few tens and few hundreds of Volts, depending obviously on the plastic used. The dependence on temperature is satisfactory, in the range of -20 ppm and 100 ppm. An important advantage of these capacitors is the maximum temperature at which they can work, ranging from 85 ^oC for usual materials to much higher temperatures for the teflon. Due to this characteristics, they are used in aerospace and other critical applications.

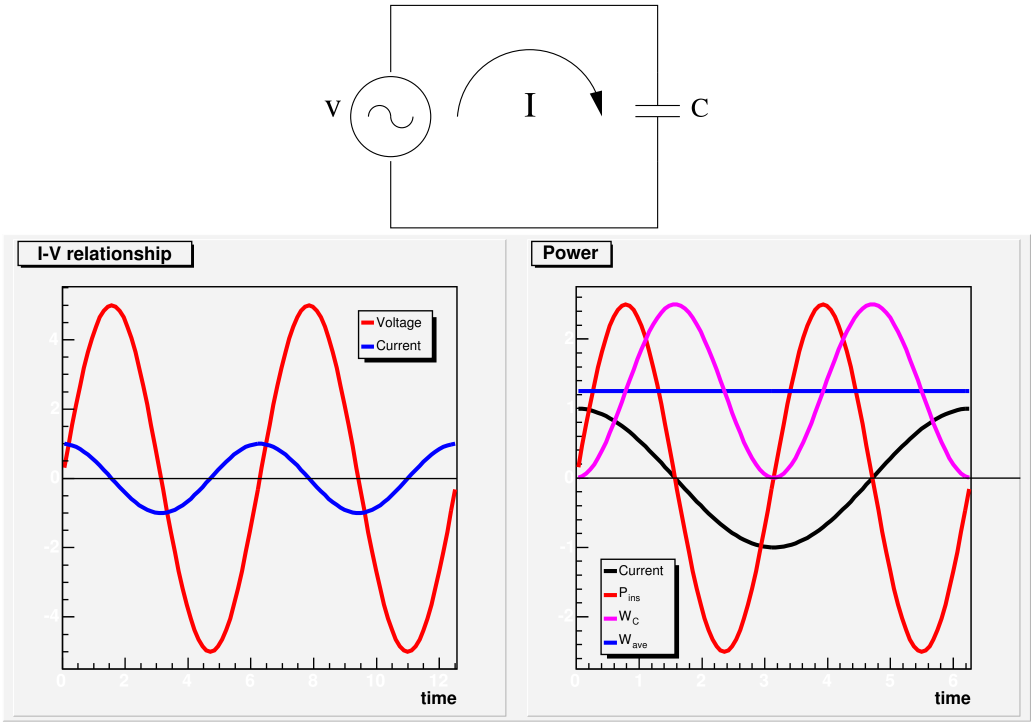


Figure 6: (a) Circuit diagram of a capacitor with a voltage V applied. (b) The current always advances the voltage by $90°$. (c) Instantaneous power P in a capacitor. Energy is stored in the capacitor when P is positive and restituted to the source for the negative case. P changes at the double of the frequency of V or I.

Ceramic capacitors They contain ceramic dielectrics, which present high dielectric constants ($\epsilon_r \sim 1200$) and high breakdown voltages. Usually they are built in multilayer monolithic or radial lead shapes, which result in small size components. These are the most popular SMD components. There are two classes of ceramic capacitors:

- class-1, ($\epsilon_r \sim 1-100$), with capacitances between few tens of nF and hundreds pF, tolerances about 2% and breakdown voltages between 100 and 500 Volts.
- class-2, ($\epsilon_r \sim 1000-10000$), with capacitances between tens pF and some μF, tolerances pretty bad, around 20% and up to 80% and breakdown voltages between 50 and 500 Volts.

Electrolytic capacitors Electrolytic capacitors have a negative (cathode) electrode, a positive (anode) one and a liquid or gel between the conducting layers. The actual dielectric is a thin oxide film on the cathode. The oxide is formed by a voltage (greater that the normal operating voltage) during manufacturing. If an electrolytic capacitor is not used for a long time, some of the oxide film gets degraded. It is reformed when voltage is applied again. Applying an excessive voltage at the capacitor causes a severe increase in leakage current, which might destroy the capacitor.

Electrolytic capacitors have voltages ranging from 6 to 450 Volts and capacitances from few tens μF to several hundreds μF at the maximum voltage, and several Farads at minimum voltage. Tolerance is quite poor and may range from $\pm$20% to $\pm^{80}_{20}$% with operating temperatures from -25 to +85 ^oC. The behaviour of these capacitors is quite poor at high frequencies. Two types of electrolytic capacitors are the most usual ones:

- Aluminum electrolytic capacitors: formed by pure aluminum foils as electrodes with an interlayer porous material (like paper) that contains the electrolyte.
- Tantalum electrolytic capacitors: the anode is a tantalum powder, the dielectric is formed by a tantalum oxide. They are the choice for small sizes.

3.3 Inductors

An inductor is a device that stores energy in the magnetic field surrounding the inductor. 'Induction' is the property of a coil (the inductor) to **oppose a changing current**. When a current flows in a stationary circuit, a magnetic field will be established with the flux density proportional to I. Therefore we can write for the flux linkage ψ

$$\psi = LI$$

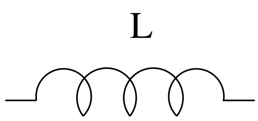


Figure 7: Schematic symbol of an inductance.

with the proportionality constant being the **self inductance** measured in $Henry\ (H)$. If the current in the circuit varies with time, and electromotive force (emf) e [V] will be induced:

$$e = -\frac{d\psi}{dt} = -L\frac{dI}{dt} = -N\frac{d\phi}{dt}$$

This relation is called **Faraday's Law of Induction**. It shows that if a conductor is placed in a time-varying magnetic field, an electromotive force is induced in the conductor which

results in a current flow. The emf is proportional to the rate of change of the magnetic flux and tends to oppose the change in current.
The magnetic flux ϕ (in $Webers\ Wb$) is related to the flux linkage via $\psi = N\phi$ (N... number of turns).

The self inductance is geometry dependent. For a coil of length l, radius r and made of N loops, the magnetic field generated by one loop is $B = \mu_0 nI\hat{z}$, where $n = N/l$. So computing the magnetic flux yields

$$\phi = \int B \cdot ds = \mu_0 nIs$$

$$L = \mu_0 ns = \frac{\mu_0 N\pi r^2}{l} = \frac{\mu_0 N\pi r^2}{d}$$

for N loops ($l = Nd$ with d = wire diameter).

Some examples:

μ_r	N	l	r	L
1	150	15 cm	25 mm	0.37 mH
1	25	5 cm	3 mm	0.4 μH
1000	10	1 cm	1 mm	39 μH

As it is shown, the inductance is proportional to the magnetic permeability. A good method to increase the inductance is to introduce a ferromagnetic material with μ_r hundreds or even thousand times greater than μ_0.

Current-Voltage relationship

We have already seen that from Faraday's law we obtain:

$$V = -L\frac{dI}{dt} \rightarrow V = -LI_p \cos t = -LI_p \sin(t + \pi/2)$$

so again voltage and current are not in phase.

Instantaneous and average power

Instantaneous power:

$$P = VI = LI\frac{dI}{dt} = \frac{LI_p^2}{2}\sin 2t$$

where again positive and negative values of the power imply that the power flows back and forth between the source and the inductor. The average power is again zero.

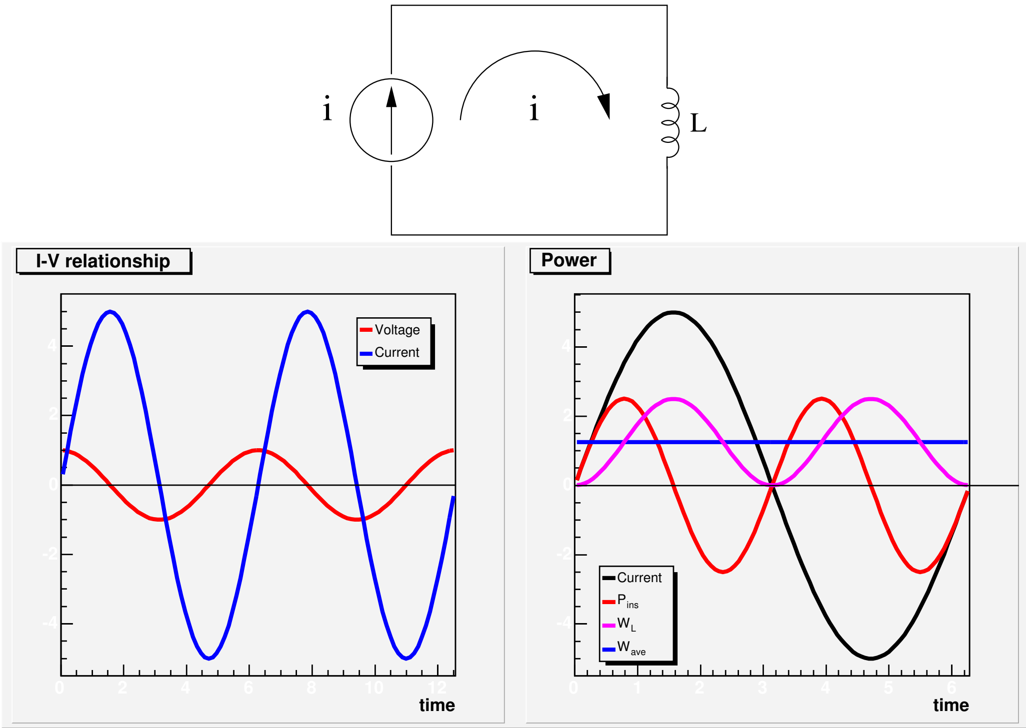


Figure 8: (a) An inductor with a voltage V applied. (b) The current is always 90° behind the voltage. (c) Instantaneous power P in an inductor. Energy is stored in the capacitor when P is positive and restituted to the source for the negative case. P changes at the double of the frequency of V or I.

Energy stored

As the current increases, a magnetic field builds up and energy is stored in the magnetic field of the inductor. When the current decreases, the stored energy of the magnetic field is released thereby adding to the current. Unlike current flowing through a resistor, energy is not dissipated in an inductor (ideally), therefore current flow through an inductor is a reversible process.
The stored energy in an inductor is given by

$$\omega_L = \int_0^t P\, dt' = \frac{1}{2} L I^2 = \frac{L I_p^2}{2} \sin^2 t$$

Therefore the average energy yields

$$\omega_{ave} = \frac{1}{T} \int_0^T \omega_L\, dt = \frac{1}{T} \frac{L I_p^2}{2} \int_0^T \sin^2 t\, dt = \frac{L I_p^2}{4}$$

4 Transformers

A transformer is a device that operates by transferring electrical energy from its input winding via a magnetic field to its output winding. The simplest transformer is made of two coils

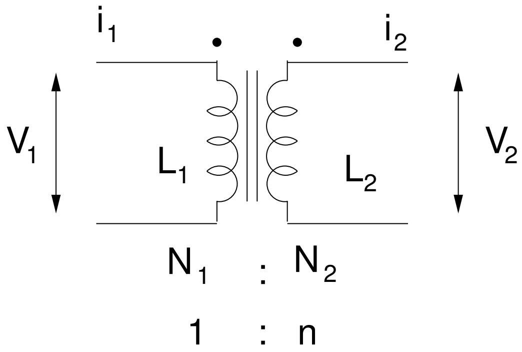


Figure 9: Schematic symbol of a transformer.

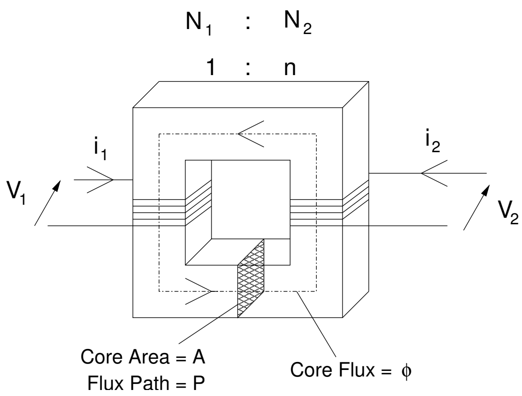


Figure 10: Ideal transformer.

close to each-other. There is no mechanical connection between the coils; they are only magnetically coupled. The windings are usually realised around a ferromagnetic material like iron or ferrite. In figure 9 the schematic symbol of such device is shown.

Transformers are used for transmitting AC power, for changing AC voltages and currents to higher or lower values, and for insulating equipment from power lines.

4.1 The ideal transformer

An ideal transformer is a transformer that has no losses. Figure 10 shows a simplified transformer with primary and secondary windings of turns ratio 1:n.

The primary applied voltage V_1 (varying in time) causes a current I_1 to flow in the primary winding, The current gives rise to a core flux ϕ which we will assume is entirely contained in the core and passes completely through the secondary windings. Thus the core flux ϕ induces a voltage V_2 across the secondary winding and a current I_2. Using Faraday's law:

$$V_1 = -N_1 \frac{d\phi}{dt}$$

$$V_2 = -N_2 \frac{d\phi}{dt}$$

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} = \frac{1}{n}$$

Only for the ideal case ($\mu \to \infty$) we can write

$$\frac{I_1}{I_2} = \frac{N_1}{N_2} = n$$

4.2 Transformer as impedance matching element

Transformers can be used for impedance matching (see next section for the definition of the impedance). In case of an ideal transformer we can write

$$\frac{Z_1}{Z_2} = \frac{V_1/I_1}{V_2/I_2} = \frac{V_1}{V_2}\frac{I_2}{I_1} = \left(\frac{N_1}{N_2}\right)^2 = \frac{1}{n^2}$$

Therefore we can easily adapt impedances between source and load of a transformer using

$$n = \sqrt{\frac{Z_{LOAD}}{Z_{SOURCE}}}$$

5 Sinusoidal analysis

Any circuit containing resistors, capacitors and inductors is linear, so if the circuit is stimulated with a sinusoidal source, all voltages and currents will also be sinusoidal.

The Fourier theorem states that any periodic function may be represented by summing sine waves of different frequencies and amplitudes. Having a periodic function $f(t)$ in a time period T defined in the interval $[0, T]$, its Fourier series is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)dt$$

where $\omega = \frac{2\pi}{T}$ and n is a postive integer number. It is important to point out that the integrals in the previous equation can be evaluated over any complete period (i.e. $[-T/2, T/2]$).

Let's suppose that we have a RLC circuit connected to a time varying voltage source $v(t)$ and we want to know what the current $i(t)$ is. Applying directly Kirchoff's voltage law yields:

$$v(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt$$

If the voltage applied is sinusoidal $v(t) = V_p \cos\omega t$, we can substitute $v(t)$ with $V_p e^{j\omega t}$ and $i(t)$ with $Ie^{j\omega t}$. We obtain (after canceling $e^{j\omega t}$):

$$V_p = RI + j\omega LI + \frac{I}{j\omega C}$$

$$V_p = \left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right] I = ZI$$

This last expression is called **generalized Ohm's Law**, and the quantity Z is the **impedance**, a complex number whose real part is the resistance R and its imaginary part is called the reactance. Simply comparing we can assign the impedance of the resistors, capacitors and inductors as:

resistor	$\rightarrow$	$Z = R$
capacitor	$\rightarrow$	$Z = \frac{-j}{\omega C}$
inductor	$\rightarrow$	$Z = j\omega L$

This is a powerful statement and means that all circuit laws derived before (Kirchoff's laws, Thevenin and Norton theorems, etc.) can be applied just substituing $v(t)$ and $i(t)$ by its complex representations and the resistance by the impedance.

The impedance takes into account the resistive and the reactive part of a circuit. Impedances are treated like vectors. The length of a vector corresponds to the magnitude of the impedance and the angle between two impedances represents the phase angle between the applied voltage and the current in the circuit. The amplitude of the total impedance in a circuit can therefore be calculated using trigonometry (Phythagoras Theorem): $Z_{tot} = \sqrt{Z_R^2 + Z_C^2 + Z_L^2}$.

6 Series and Parallel combinations of basic components

Taking into account Kirchoff's laws and applying them to the series and parallel combinations of impedances we get the following:

	Series	Parallel
Impedance	$Z_{eq} = \sum_i Z_i$	$\frac{1}{Z_{eq}} = \sum_i \frac{1}{Z_i}$
Resistors	$R_{eq} = \sum_i R_i$	$\frac{1}{R_{eq}} = \sum_i \frac{1}{R_i}$
Capacitors	$\frac{1}{C_{eq}} = \sum_i \frac{1}{C_i}$	$C_{eq} = \sum_i C_i$
Inductors	$L_{eq} = \sum_i L_i$	$\frac{1}{L_{eq}} = \sum_i \frac{1}{L_i}$

7 RC circuits

Let's analyse first the circuits in figure 11 assuming that a DC (direct current) battery of voltage V_0 is connected or disconnected at time 0 applying directly Kirchoff's laws. We consider first the case where we **charge the capacitor** (initially discharged) by switching on the DC.

$$V = V_R + V_C = Ri + \frac{q}{C} = Ri + \frac{1}{C}\int_0^t i(t)dt$$

Differentiating we obtain:

$$\frac{di}{dt} + \frac{i}{RC} = 0 \rightarrow i = Ae^{-t/RC}$$

Figure 11: Low-Pass or integration circuit on the left and High-Pass or differentiation circuit on the right.

To determine A we can use the initial condition that the capacitor was discharged ($V_C = 0$) at $t = 0$, so the current at this moment is V_0/R:

$$i(t=0) = Ae^{-0} = \frac{V_0}{R} \rightarrow i(t) = \frac{V_0}{R}e^{-t/RC}$$

At any time later, we will have that

$$V_C = \frac{q}{C} = \frac{1}{C}\int_{-\infty}^{t} i(t)dt = \frac{1}{C}\int_{0}^{t} \frac{V_0}{R}e^{-t/RC} = V_0\left(1 - e^{-t/RC}\right)$$

In case of **discharge** we consider that the DC battery is switched off and the capacitor is charged, so at $t = 0$ $V_C = V_0$. In this case we will have:

$$0 = V_R + V_C = Ri + V_0 \rightarrow i = -\frac{V_0}{R}$$

$$i(t=0) = Ae^{-0} = -\frac{V_0}{R} \rightarrow i(t) = -\frac{V_0}{R}e^{-t/RC}$$

The voltage discharge at the capacitor thus follows the following function:

$$V_C = \frac{1}{C}\int_{-\infty}^{0} i(t)dt = V_0 - \frac{1}{C}\int_{0}^{t} \frac{V_0}{R}e^{-t/RC} = V_0 e^{-t/RC}$$

7.1 RC Low-Pass Filter

Let's evaluate the output voltage characteristics with respect to the input voltage for the left circuit of fig. 11:

$$\frac{V_{out}}{Z_C} = \frac{V_{in}}{Z_C + Z_R} \rightarrow V_{out} = \frac{1/jwC}{R + 1/jwC}V_{in} = \frac{1}{1 + jwRC}V_{in}$$

The relationship between input and output voltage calculated taking into account the impedances is known as **transfer function**. This transfer function is always a function of frequency and is denoted as $H(\omega))$. For the low pass filter $H(\omega))$ is

$$H(w) = \frac{1}{1 + jwRC}$$

$$\omega \to 0 \quad \to \quad jwRC << 1 \quad \to \quad V_{out} = V_{in}$$
$$\omega \to \infty \quad \to \quad jwRC >> 1 \quad \to \quad V_{out} = (1/jwRC)V_{in}$$

As can be seen, a low-pass filter passes relatively low frequency components in the signal, but stops the high frequency components. The so-called **cut-off frequency** ω_c divides the pass band and the stop band. At this frequency, the output voltage is at 70.7% = $1/\sqrt{2}$ of the maximum value. This point is also called the 3 dB point.

$$\omega_c = \frac{1}{RC}$$

Let's consider the voltage across R:

$$I = C\frac{dV}{dt} = \frac{V_{in} - V}{R}$$

If $V << V_{in}$ (by keeping the product RC large) then:

$$C\frac{dV}{dt} \sim \frac{V_{in}}{R}$$
$$V(t) = \frac{1}{RC}\int V_{in}\, dt$$

This is the reason why a low-pass filter is also called an integration circuit.

7.2 RC High-Pass Filter

We now consider the right circuit of fig. 11:

$$\frac{V_{out}}{Z_R} = \frac{V_{in}}{Z_C + Z_R} \to V_{out} = \frac{R}{R + 1/jwC}V_{in} = \frac{jwRC}{1 + jwRC}V_{in}$$

$$H(w) = \frac{jwRC}{1 + jwRC}$$

$$\omega \to 0 \quad \to \quad jwRC << 1 \quad \to \quad V_{out} = (jwRC)V_{in}$$
$$\omega \to \infty \quad \to \quad jwRC >> 1 \quad \to \quad V_{out} = V_{in}$$

In the case of a high-pass filter, the circuit passes the high frequency components of the signal, but cuts off the low frequency components. The cut-off frequency is again $\omega_c = 1/RC$.

Looking at the voltage across C:

$$I = C\frac{d}{dt}(V_{in} - V) = \frac{V}{R}$$

If $dV/dt << dV_{in}/dt$ (keeping R and C small enough) then:

$$C\frac{dV_{in}}{dt} \sim \frac{V}{R}$$
$$V(t) = RC\frac{dV_{in}}{dt}$$

This is the reason why a high-pass filter is also called a differentiation circuit.

8 RL circuits

Figure 12: Low-Pass RL circuit on the left and High-Pass RL circuit on the right.

Let's analyse first the circuits in figure 12 assuming a DC battery of voltage V_0 that it is connected or disconnected at time 0 applying directly Kirchoff's laws. We consider first when the **battery is connected** at $t = 0$

$$V_0 = v_L + v_r = L\frac{di}{dt} + Ri \rightarrow i(t) = Ae^{-t/(L/R)} + \frac{V_0}{R}$$

At $t = 0$ the current is 0, so:

$$i(t=0) = 0 = A + \frac{V_0}{R} \rightarrow i(t) = \frac{V_0}{R}\left(1 - e^{t/(L/R)}\right)$$

Therefore v_L is:

$$v_L = L\frac{di}{dt} = V_0 e^{-t/(L/R)}$$

If we **switch off the battery**, Kirchoff's law to solve is:

$$Ri + L\frac{di}{dt} = 0 \rightarrow i(t) = Ae^{-t/(L/R)}$$

At $t = 0$ the current is V_0/R so

$$i(t=0) = \frac{V_0}{R} = A \rightarrow i(t) = \frac{V_0}{R}e^{-t/(L/R)}$$

so the voltage across L is:

$$v_L = L\frac{di}{dt} = -V_0 e^{-t/(L/R)}$$

8.1 RL Low-pass filter

$$V_{out} = \frac{Z_R}{Z_L + Z_R}V_{in} = \frac{R}{R + jwL}V_{in} = \frac{1}{1 + jwL/R}V_{in}$$

$$H(w) = \frac{1}{1 + jwL/R}$$

$$\begin{array}{lllll} \omega \rightarrow 0 & \rightarrow & jwL/R << 1 & \rightarrow & V_{out} = V_{in} \\ \omega \rightarrow \infty & \rightarrow & jwL/R >> 1 & \rightarrow & V_{out} = 1/jw(L/R)V_{in} \end{array}$$

$$\omega_c = R/L$$

8.2 RL High-pass filter

$$V_{out} = \frac{Z_L}{Z_L + Z_R} V_{in} = \frac{jwL}{R + jwL} V_{in} = \frac{jwL/R}{1 + jwL/R} V_{in}$$

$$H(w) = \frac{jwL/R}{1 + jwL/R}$$

$$\omega \to 0 \quad \to \quad jwL/R << 1 \quad \to \quad V_{out} = jw(L/R)V_{in}$$

$$\omega \to \infty \quad \to \quad jwL/R >> 1 \quad \to \quad V_{out} = V_{in}$$

$$\omega_c = R/L$$

9 n-pole filters

One can make circuits with several RC sections. E.g. for a simple RC filter the output amplitude will drop to half or 6 dB in the falloff region of the signal when the frequency doubles (after one 'octave'). Adding another RC chain will yield 12 dB per octave, another one 16 dB/octave. Such filters are called 'n-pole filters' with n being the number of RC filters (or filters that behave like them).

10 RCL circuits; Resonance

When capacitors are combined with inductors or are used in special circuits called active filters, it is possible to make circuits that have very sharp frequency characteristics (e.g. a large peak in the response at a particular frequency). These circuits find applications in various audiofrequency and radiofrequency devices.

10.1 Serial circuit. Band elimination circuit

Fig. 13 (left) shows a basic band-elimination resonant circuit. The output voltage for this type of filter is taken at the LC serial combination. As we will see, such a circuit will cut out a certain frequency from the signal. Frequencies below the low frequency cut ω_- and frequencies above the high frequency cut ω_+ are transferred (with an attenuation $\leq$3 dB of the maximum output signal). At the resonance frequency ω_0, ideally the output signal goes to zero.

$$V_{in} = V_p \cos wt \to I = I_p e^{-j\theta}$$

$$I_p = \frac{V_p}{\sqrt{R^2 + (wL - 1/wC)^2}}$$

$$\theta = \tan^{-1} \frac{wL - 1/wC}{R}$$

$$Z = R + j(wL - 1/wC)$$

$$V_{out} = \frac{Z_C + Z_L}{Z_R + Z_C + Z_L} V_{in} = \frac{j(wL - 1/wC)}{R + j(wL - 1/wC)} V_{in}$$

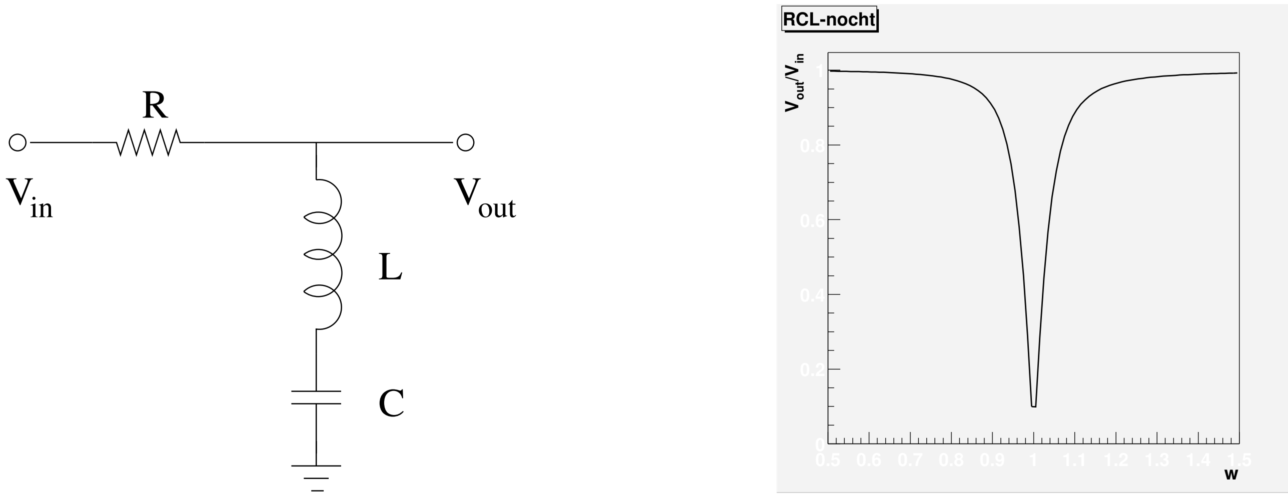


Figure 13: A band elimination or notch filter and the voltage gain response as a function of frequency.

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{(wL - 1/wC)}{\sqrt{R^2 + (wL - 1/wC)^2}}$$

Resonance is defined as the condition when current and voltage are in phase. In the case we are studying, resonance is at $\omega_0 = 1/\sqrt{LC}$ (where $\omega L = 1/\omega C$). At this frequency we will have the following situation:

- The current has a maximum $I_p = V_p/R$.
- The impedance has a minimum $Z = R$. The impedance of LC goes to zero; signals are 'trapped' near at or near the resonance frequency, shorting them to ground.
- The output voltage has a minimum $V_{out} = 0$.

10.1.1 Quality factor and bandwidth

The quality factor Q is a measure of the sharpness of the resonance peak. It equals the resonant frequency divided by the width at the -3 dB points. The larger Q, the sharper the peak. Q is defined as the ratio between the reactive power divided by the real power:

$$Q = \frac{I^2 Z}{I^2 R}$$

In a serial resonance circuit at the resonance frequency (where $Z_L = Z_C$), Q is therefore given by

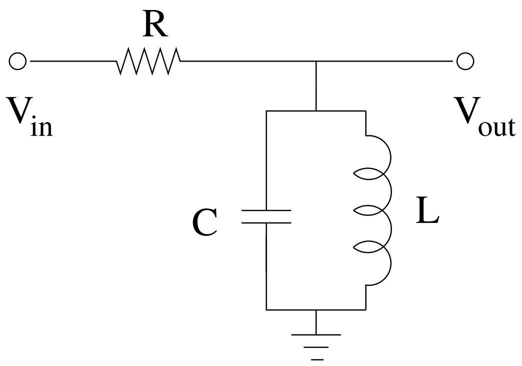
$$Q = \frac{Z_L}{R} = \frac{\omega L}{R}$$

Geometrically, the quality factor can be determined by measuring ω_- and ω_+ at $\pm 1/\sqrt{2}$ of the maximum output signal height. Q is then given by

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{\omega_+ - \omega_-}$$

The frequency band between ω_- and ω_+ is called **bandwidth.**

10.2 Parallel circuit. Band pass filter



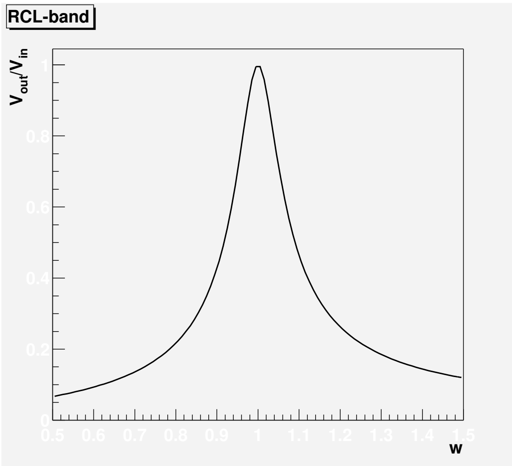


Figure 14: Parallel RCL

$$Z = Z_R + Z_{LC} = Z_R + (Z_L^{-1} + Z_C^{-1})^{-1} = R + \frac{j}{1/wL - wC}$$

$$I_p = \frac{V_p}{\sqrt{R^2 + 1/(1/wL - wC)^2}}$$

$$\theta = \tan^{-1}\left(\frac{1/wL - wC}{R}\right)$$

$$V_{out} = \frac{Z_{LC}}{Z_R + Z_{LC}} V_{in} = \frac{1}{1 - jR(1/wL - wC)} V_{in}$$

$$\left|\frac{V_{out}}{V_{in}}\right| == \frac{1}{\sqrt{1 + R^2(wC - 1/wL)^2}}$$

Again the resonance frequency is at $\omega_0 = 1/\sqrt{LC}$. In this case a narrow band of frequencies is passed through the circuit, cutting of the low and high frequency parts. At the resonance frequency we have:

- Minimum $I_p = V_p/R$.
- Maximum impedance of the parallel LC (goes to infinity due to the opposite behaviours of inductors and capacitors).
- Maximum $V_{out} = V_{in}$.

10.2.1 Quality factor

For a parallel RLC circuit, $Q = \omega_0 RC = \omega_0/\Delta\omega$.

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